18 The Exterior Angle Theorem and Its Consequences

Definition. **(less than & greater than for line segments)** In a metric geometry, the line segment $\overline{AB}$ is less than (or smaller than) the line segment $\overline{CD}$ (written $\overline{AB} < \overline{CD}$) if $AB < CD$. $\overline{AB}$ is greater than (or larger than) $\overline{CD}$ if $AB > CD$. The symbol $\overline{AB} \leq \overline{CD}$ means that either $\overline{AB} < \overline{CD}$ or $\overline{AB} \cong \overline{CD}$.

Definition. **(less than & greater than for angles)** In a protractor geometry, the angle $\measuredangle ABC$ is less than (or smaller than) the angle $\measuredangle DEF$ (written $\measuredangle ABC < \measuredangle DEF$) if $m(\measuredangle ABC) < m(\measuredangle DEF)$. $\measuredangle ABC$ is greater than (or larger than) $\measuredangle DEF$ if $\measuredangle DEF < \measuredangle ABC$). The symbol $\measuredangle ABC \leq \measuredangle DEF$) means that either $\measuredangle ABC < \measuredangle DEF$) or $\measuredangle ABC \cong \measuredangle DEF$).

Theorem. In a metric geometry, $\overline{AB} < \overline{CD}$ if and only if there is a point $G \in \text{int}(\overline{CD})$ with $\overline{AB} \cong \overline{CG}$.

1. Prove the above Theorem.

Theorem. In a protractor geometry, $\measuredangle ABC < \measuredangle DEF$) if and only if there is a point $G \in \text{int}(\measuredangle DEF)$ with $\measuredangle ABC \cong \measuredangle DEG$).

2. Prove the above Theorem.

Definition. **(exterior angle, remote interior angle)** Given $\triangle ABC$ in a protractor geometry, if $A{-}C{-}D$ then $\measuredangle BCD$ is an exterior angle of $\triangle ABC$. $\measuredangle A$ and $\measuredangle B$ are the remote interior angles of the exterior angle $\measuredangle BCD$.

Theorem (Exterior Angle Theorem). In a neutral geometry, any exterior angle of $\triangle ABC$ is greater than either of its remote interior angles.

3. Prove the above Theorem. [Th 6.3.3, p. 136]

4. In a protractor geometry prove the two exterior angles of $\triangle ABC$ at the vertex C are congruent.

5. In a neutral geometry prove that the base angles of an isosceles triangle are acute.

6. Show that at most one angle in triangle can be right or obtuse angle, and that at least two angles are acute.

Corollary In a neutral geometry, there is exactly one line through a given point P perpendicular to a given line ℓ.

7. Prove the above Corollary. [Cor 6.3.4, p. 137]

Theorem (Side-Angle-Angle, SAA). In a neutral geometry, given two triangles $\triangle ABC$ and $\triangle DEF$, if $\overline{AB} \cong \overline{DE}$, $\measuredangle A \cong \measuredangle D$, and $\measuredangle C \cong \measuredangle F$, then $\triangle ABC \cong \triangle DEF$.

8. Prove the above Theorem. [Th 6.3.5, p 138]

We should note that the above proof (which is valid in any neutral geometry) is probably different from any you have seen before. In particular we did not prove $\measuredangle B \cong \measuredangle E$ by looking at the sums of the measures of the angles of the two triangles. We could not do this because we do not know any theorems about the sum of the measures of the angles of a triangle. **In particular the sum may not be the same for two triangles in an arbitrary neutral geometry**.

Theorem In a neutral geometry, if two sides of a triangle are not congruent, neither are the opposite angles. Furthermore, the larger angle is opposite the longer side.

9. Prove the above Theorem. [Th 6.3.6, p 138]

Theorem In a neutral geometry, if two angles of a triangle are not congruent, neither are the opposite sides. Furthermore, the longer side is opposite the larger angle.

10. Prove the above Theorem.

Theorem (Triangle Inequality). In a neutral geometry the length of one side of a triangle is strictly less than the sum of the lengths of the other two sides.

11. Prove the above Theorem. [Th 6.3.8, p 139]

12. In a neutral geometry, if $D \in \text{int}(\triangle ABC)$ prove that $AD + DC < AB + BC$ and $\measuredangle ADC > \measuredangle ABC$.

Theorem (Open Mouth Theorem). In a neutral geometry, given two triangles $\triangle ABC$ and $\triangle DEF$ with $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, if $\measuredangle B > \measuredangle E$ then $\overline{AC} > \overline{DF}$.

13. Prove the above Theorem. [Th 6.3.9, p 140]

Theorem In a neutral geometry, a line segment joining a vertex of a triangle to a point on the opposite side is shorter than the longer of the remaining two sides. More precisely, given $\triangle ABC$ with $\overline{AB} \leq \overline{CB}$, if $A - D - C$ then $\overline{DB} < \overline{CB}$.

14. Prove the above Theorem.

15. Prove the converse of Open Mouth Theorem: In a neutral geometry, given $\triangle ABC$ and $\triangle DEF$, if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ and $\overline{AC} > \overline{DF}$, then $\measuredangle B > \measuredangle E$.

16. In a neutral geometry, given $\triangle ABC$ such that the internal bisectors of $\measuredangle A$ and $\measuredangle C$ are congruent, prove that $\triangle ABC$ is isosceles.

17. Replace the word "neutral" in the hypothesis of Theorem 6.3.6 (Problem 9) with the word "protractor". Is the conclusion still valid?

19 Right Triangles

A word of caution is needed here. The first thing that many of us think about when we hear the phrase "right triangle" is the classical **Pythagorean Theorem**. This theorem is very much a Euclidean theorem. That is, it **is true in the Euclidean Plane but not in all neutral geometries** (see Problem 10). Thus in each proof of this section which deals with a general neutral geometry we must avoid the use of the Pythagorean Theorem.

Definition. **(right triangle, hypotenuse)** If an angle of $\triangle ABC$ is a right angle, then $\triangle ABC$ is a right triangle. A side opposite a right angle in a right triangle is called a hypotenuse.

Definition. **(the longest side, a longest side)** $\overline{AB}$ is the longest side of $\triangle ABC$ if $\overline{AB} > \overline{AC}$ and $\overline{AB} > \overline{BC}$. $\overline{AB}$ is a longest side of $\triangle ABC$ if $\overline{AB} \geq \overline{AC}$ and $\overline{AB} \geq \overline{BC}$.

Theorem In a neutral geometry, there is only one right angle and one hypotenuse for each right triangle. The remaining angles are acute, and the hypotenuse is the longest side of the triangle.

1. Prove the above Theorem. [Th 6.4.1, p 143]

Definition. **(legs)** If $\triangle ABC$ is a right triangle with right angle at C then the legs of $\triangle ABC$ are $\overline{AC}$ and $\overline{BC}$.

Theorem (Perpendicular Distance Theorem). In a neutral geometry, if ℓ is a line, $Q \in \ell$, and $P \notin \ell$ then (i) if $\overleftrightarrow{PQ} \perp \ell$ then $PQ \leq PR$ for all $R \in \ell$ (ii) if $PQ \leq PR$ for all $R \in \ell$ then $\overleftrightarrow{PQ} \perp \ell$.

2. Prove the above Theorem. [Th 6.4.2, p 144]

Definition. **(distance from P to ℓ)** Let ℓ be a line and P a point in a neutral geometry. If $P \notin \ell$, let Q be the unique point of ℓ such that $\overleftrightarrow{PQ} \perp \ell$. The distance from P to ℓ is

$$d(P,\ell) = \begin{cases} d(P,Q), & \text{if } P \notin \ell \\ 0, & \text{if } P \in \ell. \end{cases}$$

Theorem For any line ℓ in a neutral geometry and $P \notin \ell$ $\quad d(P,\ell) \leq d(P,R)$ for all $R \in \ell$. Furthermore, $d(P,\ell) = d(P,R)$ if and only if $\overleftrightarrow{PR} \perp \ell$.

Definition. **(altitude, foot of the altitude)** If ℓ is the unique perpendicular to $\overleftrightarrow{AB}$ through the vertex C of $\triangle ABC$ and if $\ell \cap \overleftrightarrow{AB} = \{D\}$, then $\overline{CD}$ is the altitude from C. D is the foot of the altitude (or of the perpendicular) from C.

Theorem In a neutral geometry, if $\overline{AB}$ is a longest side of $\triangle ABC$ and if D is the foot of the altitude from C, then $A - D - B$.

3. Prove the above Theorem. [Th 6.4.3, p 145]

Theorem (Hypotenuse-Leg, HL). In a neutral geometry if $\triangle ABC$ and $\triangle DEF$ are right triangles with right angles at C and F, and if $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$, then $\triangle ABC \cong \triangle DEF$.

4. Prove the above Theorem. [Th 6.4.4, p 146]

Theorem (Hypotenuse-Angle, HA). In a neutral geometry, let $\triangle ABC$ and $\triangle DEF$ be right triangles with right angles at C and F. If $\overline{AB} \cong \overline{DE}$ and $\measuredangle A \cong \measuredangle D$, then $\triangle ABC \cong \triangle DEF$.

Definition. **(perpendicular bisector)** The perpendicular bisector of the segment $\overline{AB}$ in a neutral geometry is the (unique) line ℓ through the midpoint M of $\overline{AB}$ and which is perpendicular to $\overline{AB}$.

Theorem In a neutral geometry the perpendicular bisector ℓ of the segment $\overline{AB}$ is the set $\mathcal{B} = \{P \in \mathcal{S} \mid AP = BP\}$.

5. Prove the above Theorem. [Th 6.4.6, p 147]

6. In a neutral geometry, if D is the foot of the altitude of $\triangle ABC$ from C and $A - B - D$, then prove $\overline{CA} > \overline{CB}$.

7. In a neutral geometry, denote by M_1 the foot of the altitude of $\triangle ABM$ from M and let $A - M_1 - B$. Prove that then $\overline{MA} > \overline{MB}$ if and only if $\overline{M_1A} > \overline{M_1B}$.

8. If M is the midpoint of $\overline{BC}$ then $\overline{AM}$ is called a **median** of $\triangle ABC$. Consider $\triangle ABC$ such that $\overline{AB} < \overline{AC}$. Let E, D and H denote the points in which bisector of angle, median and altitude from A intersect line $\overleftrightarrow{BC}$, respectively. Show that (a) $\measuredangle AEB < \measuredangle AEC$; (b) $\overline{BE} < \overline{CE}$; (c) we have $H - E - D$.

9. (a.) Prove that in a neutral geometry if $\triangle ABC$ is isosceles with base $\overline{BC}$ then the following are collinear: (i) the median from A; (ii) the bisector of $\measuredangle A$; (iii) the altitude from A; (iv) the perpendicular bisector of $\overline{BC}$. (b.) Conversely, in a neutral geometry prove that if any two of (i)-(iv) are collinear then the triangle is isosceles (six different cases).

10. Show that the conclusion of the Pythagorean Theorem is not valid in the Poincaré Plane by considering $\triangle ABC$ with $A(2,1)$, $B(0,\sqrt{5})$, and $C(0,1)$. Thus the Pythagorean Theorem does not hold in every neutral geometry.

Theorem In a neutral geometry, if $\overrightarrow{BD}$ is the bisector of $\measuredangle ABC$ and if E and F are the feet of the perpendiculars from D to $\overleftrightarrow{BA}$ and $\overleftrightarrow{BC}$ then $\overline{DE} \cong \overline{DF}$.

11. Prove the above Theorem. [Th 6.4.7, p 148]

Teorema

U metričkoj geometriji $\overline{AB} < \overline{CD}$ ako i samo ako postoji tačka $G \in int(\overline{CD})$ taka da $\overline{AB} \cong \overline{CG}$.

\# Dokazati teoremu iznad.

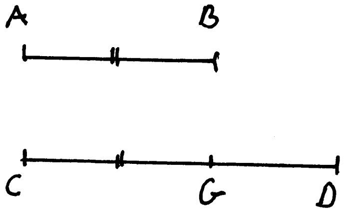
Rj.

„⇐": Pretpostavimo da $\exists G \in int(\overline{CD})$ taka da $\overline{AB} \cong \overline{CG}$ i pokažimo da $\overline{AB} < \overline{CD}$.

Prisjetimo se

$$\overline{CD} \overset{def.}{=} \{M \in \mathcal{P} \mid C-M-D \text{ ili } C=M \text{ ili } D=M\}$$

$$int(\overline{CD}) \overset{def.}{=} \overline{CD} - \{C, D\} = \{M \in \mathcal{P} \mid C-M-D\}$$



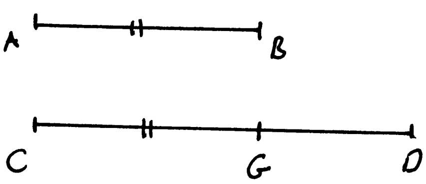
$$G \in int(\overline{CD}) \Rightarrow C-G-D \Rightarrow CD = CG + GD \quad \dots (1)$$

$$\overline{AB} \cong \overline{CG} \Rightarrow AB = CG \quad \dots (2)$$

$$(1) \text{ i } (2) \Rightarrow CD = AB + GD \Rightarrow AB < CD \Rightarrow \overline{AB} < \overline{CD}$$

q.e.d.

„⇒": Pretpostavimo da je $\overline{AB} < \overline{CD}$ i pokažimo da $\exists G \in int(\overline{CD})$ t.d. $\overline{AB} \cong \overline{CG}$.



Prisjetimo se teoreme konstrukcije duži!
Ako je $\overrightarrow{AB}$ data poluprava i $\overline{PQ}$ duž u metričkoj geometriji tada postoji jedinstvena tačka $C \in \overrightarrow{AB}$ taka da $\overline{PQ} \cong \overline{AC}$

Prema teoremu konstrukcije duži za $\overrightarrow{CD}$ i $\overline{AB}$ $\exists! G \in \overrightarrow{CD}$ t.d. $\overline{AB} \cong \overline{CG}$

$$\left.\begin{array}{l} \overline{AB} \cong \overline{CG} \Rightarrow AB = CG \\ \overline{AB} < \overline{CD} \Rightarrow AB < CD \end{array}\right\} \Rightarrow CG < CD \Rightarrow C-G-D \Rightarrow G \in int(\overline{CD})$$

Time smo pokazali da $\exists G \in int(\overline{CD})$ t.d. $\overline{AB} \cong \overline{CG}$

q.e.d.

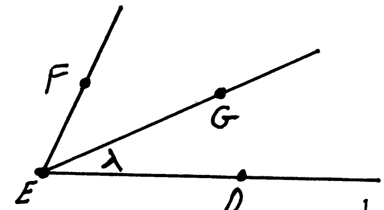
Teorema

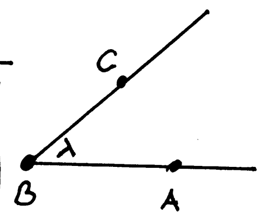
U protractor geometriji $\measuredangle ABC < \measuredangle DEF$ ako i samo ako postoji tačka $G \in int(\measuredangle DEF)$ takva da $\measuredangle ABC \cong \measuredangle DEG$.

(#) Dokazati teoremu iznad.

Rj.

„$\Leftarrow$" Pretpostavimo da postoji tačka $G \in int(\measuredangle DEF)$ takva da $\measuredangle ABC \cong \measuredangle DEG$ (i pokažimo da je tada $\measuredangle ABC < \measuredangle DEF$).





Prisjetimo se
$\measuredangle DEF \stackrel{def}{=} \overrightarrow{ED} \cup \overrightarrow{EF} = pp[E,D) \cup pp[E,F)$
$int(\measuredangle DEF) \stackrel{def}{=}$ presjek strane prave $\overleftrightarrow{ED}$ koja sadrži tačku F sa stranom prave $\overleftrightarrow{EF}$ koja sadrži tačku D

Znamo $G \in int(\measuredangle DEF)$ akko G i F su sa iste strane $\overleftrightarrow{ED}$ i G i D su sa iste strane $\overleftrightarrow{EF}$

$\Rightarrow \quad m(\measuredangle DEF) = m(\measuredangle DEG) + m(\measuredangle GEF) \quad \dots(1)$

$\measuredangle ABC \cong \measuredangle DEF \Rightarrow m(\measuredangle ABC) = m(\measuredangle DEF) \quad \dots(2)$

(1) i (2) $\Rightarrow m(\measuredangle DEF) = m(\measuredangle ABC) + \underbrace{m(\measuredangle GEF)}_{>0} \Rightarrow m(\measuredangle ABC) < m(\measuredangle DEF)$
$\Downarrow$
$\measuredangle ABC < \measuredangle DEF$
q.e.d.

„$\Rightarrow$" Pretpostavimo da je $\measuredangle ABC < \measuredangle DEF$
(i pokažimo da je tada $\exists G \in int(\measuredangle DEF)$ t.d. $\measuredangle ABC \cong \measuredangle DEG$)

Prisjetimo se teoreme konstrukcije ugla.
U protractor geometriji, za dati ugao $\measuredangle ABC$ i polupravu $\overrightarrow{ED}$ koja pripada ivici poluravni H_1, postoji jedinstvena poluprava $\overrightarrow{EG}$ takva da $G \in H_1$ i $\measuredangle ABC \cong \measuredangle DEG$

Označimo sa H poluravan sa ivicom u pravoj $\overleftrightarrow{ED}$ koja sadrži tačku F.
Prema Teoremu konstrukcije ugla $\exists! \overrightarrow{EG}$ t.d. $G \in H$ i $\measuredangle ABC \cong \measuredangle DEG$.

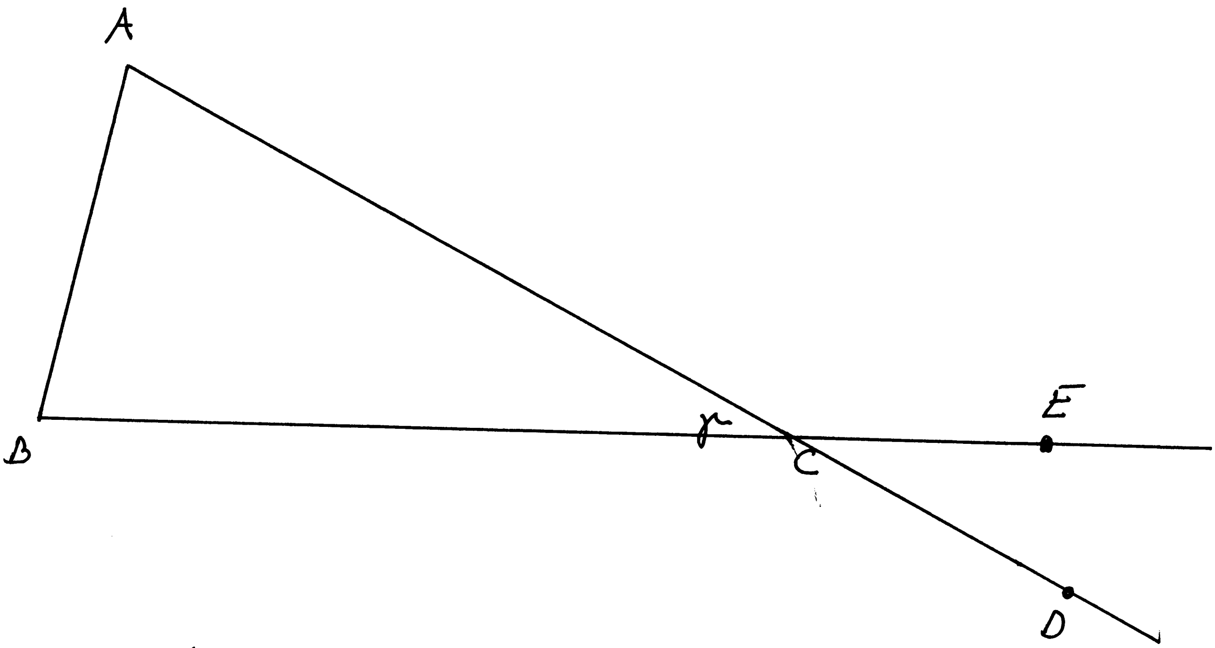
$$\left.\begin{array}{l} \measuredangle ABC < \measuredangle DEF \Rightarrow m(\measuredangle ABC) < m(\measuredangle DEF) \\ \measuredangle ABC \cong \measuredangle DEF \Rightarrow m(\measuredangle ABC) = m(\measuredangle DEG) \end{array}\right\} \Rightarrow m(\measuredangle DEG) < m(\measuredangle DEF) \Rightarrow G \in int(\measuredangle DEF)$$

Time smo pokazali da $\exists G \in int(\measuredangle DEF)$ t.d. $\measuredangle ABC \cong \measuredangle DEG$.

(#) U Protractor geometriji pokazati da su dva vanjska vrha trougla ΔABC na vrhu C podudarna.

Rj.

Neka je ΔABC dati trougao i pretpostavimo da je $A-C-D$ i $B-C-E$. Dva ugla koja posmatramo su $\measuredangle BCD$ i $\measuredangle ACE$. Označimo sa γ mjeru ugla $\measuredangle ACB$.



Primjetimo da uglovi $\measuredangle BCA$ i $\measuredangle ACE$ formiraju linearan par $\Rightarrow$

$$\Rightarrow \gamma + m(\measuredangle ACE) = 180 \Rightarrow m(\measuredangle ACE) = 180 - \gamma \quad \ldots(1)$$

Slično, uglovi $\measuredangle ACB$ i $\measuredangle BCD$ formiraju linearan par

$$\gamma + m(\measuredangle BCD) = 180 \Rightarrow m(\measuredangle BCD) = 180 - \gamma \quad \ldots(2)$$

$$(1) \text{ i } (2) \Rightarrow m(\measuredangle ACE) = m(\measuredangle BCD) \Rightarrow \measuredangle ACE \cong \measuredangle BCD$$

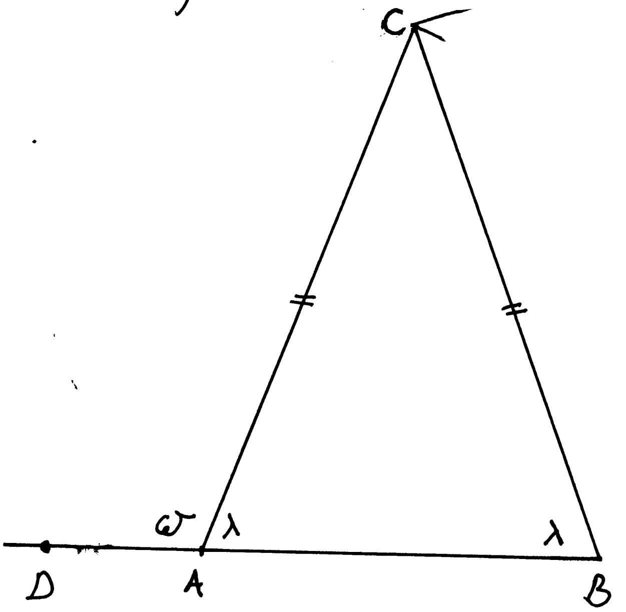
q.e.d.

U neutralnoj geometriji pokazati da su uglovi na osnovici jednakokrakog trougla oštri.

Rj.

Za ugao kažemo da je oštar ako je njegova mjera manja od 90.

Neka je dat jkk $\triangle ABC$ u kojem je $AC \cong BC$.



Znamo da je tada $\measuredangle CAB \cong \measuredangle CBA$. Označimo mjere ova dva ugla sa λ.

Neka je D tačka na pravoj $\overleftrightarrow{AB}$ t.d. $B-A-D$.

Primjetimo da je $\measuredangle CAD$ vanjski ugao $\triangle ABC$ na vrhu A. Mjeru ugla ovog trougla označimo sa ω.

Prema teoremu vanjskog ugla imamo da je $\measuredangle CAD > \measuredangle CBA$ tj.

$$\omega > \lambda. \quad \ldots(*)$$

Podjelimo dokaz ovog zadatka na tri slučaja

$1^\circ\ \lambda > 90$

$2^\circ\ \lambda = 90$ i pokažimo da slučajevi 1 i 2 nisu mogući.

$3^\circ\ \lambda < 90$

Ako bi bilo $1^\circ\ \lambda > 90$, s obzirom da uglovi $\measuredangle DAC$ i $\measuredangle CAB$ formiraju linearan par imamo da $\omega + \lambda = 180 \overset{\lambda>90}{\Longrightarrow} \omega < 90 \Longrightarrow \omega < \lambda$

#kontradikcija (sa $(*)$)

Ako bi bilo $2^\circ\ \lambda = 90$, s obzirom da uglovi $\measuredangle DAC$ i $\measuredangle CAB$ formiraju linearan par imamo da $\omega + \lambda = 180^\circ \overset{\lambda=90}{\Longrightarrow} \omega = 90 \Longrightarrow \omega = \lambda$

#kontradikcija (sa $(*)$).

Prema tome mora vrijediti $3^\circ\ \lambda < 90$

$\Longrightarrow$ uglovi na osnovici su oštri.

(#) Pokazati da najviše jedan ugao u trouglu može biti prav ili tup, a najmanje dva su oštra.

Rj.

Pretpostavimo suprotno tvrdnji tj. pretpostavimo da vrijedi jedan od sljedećih slučajeva

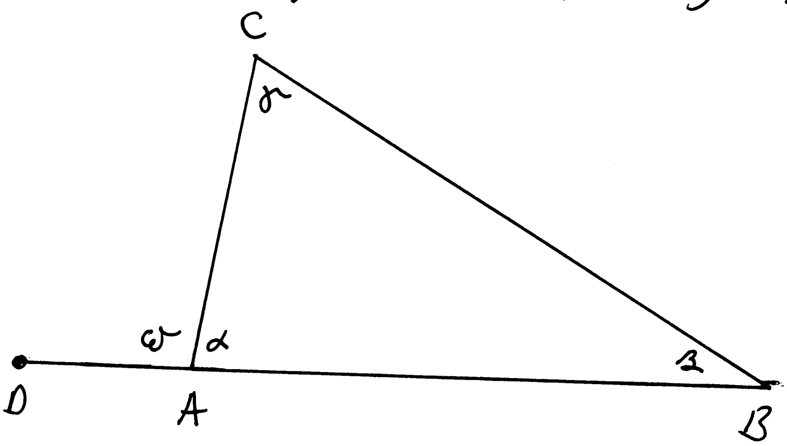
(a) dva ugla su prava

(b) jedan je prav, jedan tup

(c) dva ugla su tupa.

Pokažimo da slučaj (a) nije moguć. Slično se pokazuju slučajevi (b) i (c).

Neka je dat $\triangle ABC$, i pretpostavimo da je $\measuredangle BAC$ prav ugao. Označimo mjere uglova $\measuredangle A$, $\measuredangle B$ i $\measuredangle C$ redom sa α, β i γ.



Neka je $D \in pp[B, A) = \overrightarrow{BA}$ t.d. $B-A-D$. Mjeru ugla $\measuredangle CAD$ označimo sa ω.

Prema teoremu vanjskog ugla primjećujemo da je $\omega > \beta$ i $\omega > \gamma$. ...(1)

Kako je $\alpha + \omega = 180$ i $\alpha = 90$ to je $\omega = 90$.

Ako bi ugao β (ili ugao γ) bio prav ugao tada bi dobili da je $\omega = \beta$ (ili $\omega = \gamma$) što je u kontradikciji sa (1).

Nešto slično bi imali da smo pretpostavili da su β i γ pravi uglovi.

Prema tome najviše jedan ugao u trouglu može biti prav ili tup, a najmanje dva su oštra.

Teorema

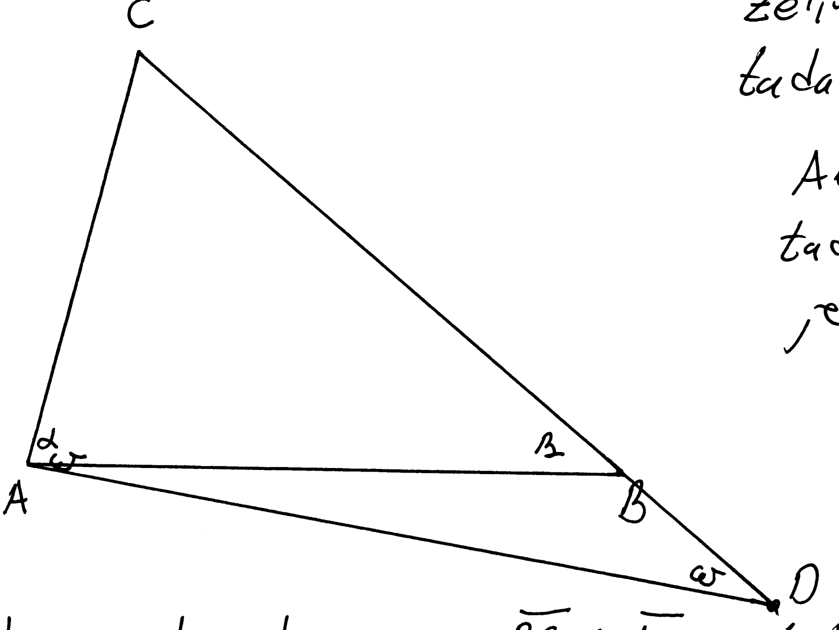
U neutralnoj geometriji, ako dva ugla trougla nisu podudarna, tada nisu podudarne ni odgovarajuće nasuprotne stranice. Štaviše, veća strana je nasuprot većeg ugla.

(#) Dokazati teoremu iznad

Rj: U trouglu ΔABC pretpostavimo da je $\sphericalangle A > \sphericalangle B$.

Želimo pokazati da je tada $\overline{BC} > \overline{AC}$.

Ako bi bilo $\overline{BC} \cong \overline{AC}$ tada bi dobili da je $\sphericalangle CAB \cong \sphericalangle CBA$
#kontradikcija (sa pretpostavkom da je $\sphericalangle A > \sphericalangle B$)



Ako bi imali da je $\overline{BC} < \overline{AC}$ tada $\exists D \in [C,B) = \overrightarrow{CB}$ t.d. $C-B-D$ i $\overline{AC} \cong \overline{CD}$ (ZAŠTO?)

Kako je $C-B-D \Rightarrow B \in int(\sphericalangle CAD) \Rightarrow \sphericalangle CAB$

Meru ugla $\sphericalangle CAD$ označimo sa ω (time imamo da je $\alpha < \omega$).

Sad primjetimo da je $\sphericalangle ABC$ vanjski ugao ΔADB i primjetimo da je $\sphericalangle CAD \cong \sphericalangle ADC$. Time imamo

$$\beta > \omega > \alpha \Rightarrow \sphericalangle B > \sphericalangle A$$

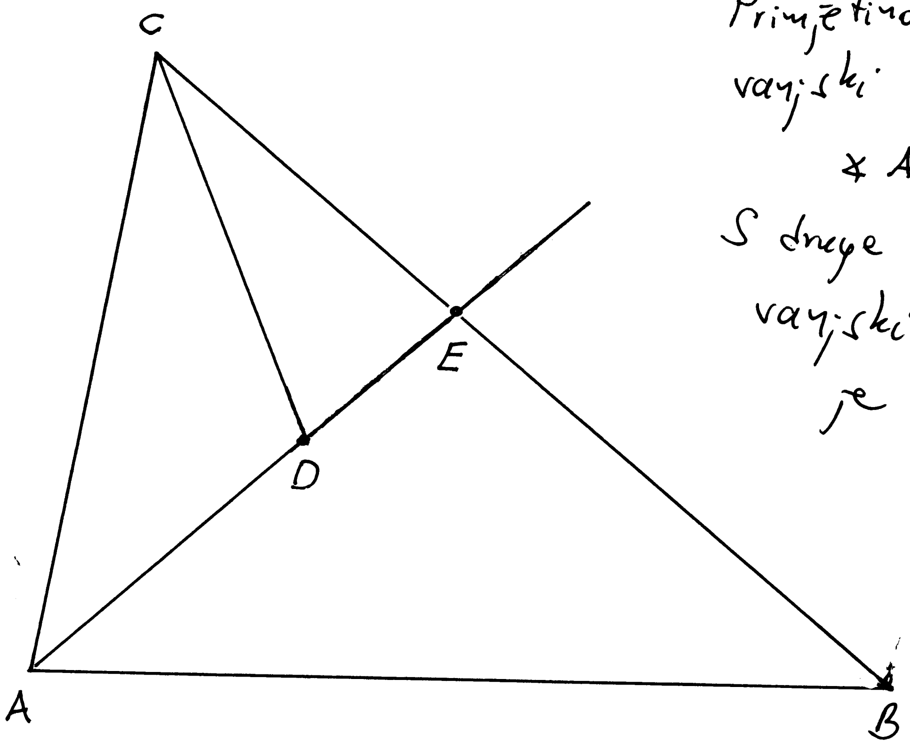
#kontradikcija (prema pretpostavci $\sphericalangle A > \sphericalangle B$).

Pretpostavka da je $\overline{BC} \leq \overline{AC}$ nas vodi u kontradikciju pa nije tačna. Prema tome mora vrijediti da je $\overline{BC} > \overline{AC}$.

U neutralnoj geometriji, ako $D \in \operatorname{int}(\measuredangle ABC)$ pokazati da

$$AD + DC < AB + BC \quad i \quad \measuredangle ADC > \measuredangle ABC.$$

Rj.



Primjetimo da je $\measuredangle ADC$ vanjski ugao $\triangle CDE$ pa je

$$\measuredangle ADC > \measuredangle DEC \quad \dots(1)$$

S druge strane $\measuredangle DEC$ je vanjski ugao $\triangle ABE$ pa je

$$\measuredangle AEC > \measuredangle ABE \quad \dots(2)$$

Na osnovu (1) i (2)

$$\measuredangle ADC > \measuredangle ABC.$$

q.e.d.

$$CD < CE + DE$$

$$+ \quad AD + DE < AB + BE$$

$$AD + \cancel{DE} + CD < CE + \cancel{DE} + AB + BE$$

$$AD + DC < AB + BC$$

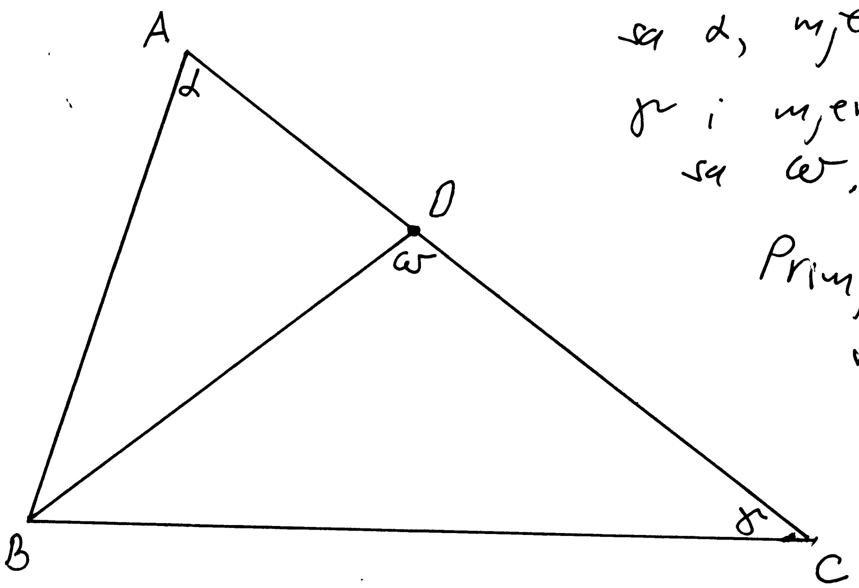
q.e.d.

Teorem

U neutralnoj geometriji, duž koja spaja vrh trougla sa tačkom na suprotnoj strani je kraća od duže od preostale dvije strane. Preciznije, za dati trougao $\triangle ABC$ takav da $\overline{AB} \leq \overline{CB}$, ako je $A-D-C$ tada je $\overline{DB} < \overline{CB}$.

(#) Dokazati teorem iznad.

R: Neka je dat $\triangle ABC$ t.d. $\overline{AB} \leq \overline{CB}$ i neka je D tačka na stranici $\overline{AC}$ t.d. $C-D-A$. Mjeru ugla $\measuredangle A$ označimo sa α, mjeru ugla $\measuredangle C$ označimo sa γ i mjeru ugla $\measuredangle BDC$ označimo sa ω.



Primjetimo da je $\measuredangle BDC$ vanjski ugao $\triangle ABD$ pa je

$$\omega > \alpha \quad , \quad \ldots (*)$$

Ako bi vrijedilo da je $\overline{BD} \cong \overline{CB}$ tada bi $\triangle BCD$ bio jkk pa bi imali da je $\omega = \gamma$. Na osnovu $(*)$ ovo povlači da je $\gamma > \alpha \overset{\triangle ABC}{\Longrightarrow} \overline{AB} > \overline{BC}$

#kontradikcija
(prema pretpostavci $\overline{AB} \leq \overline{CB}$)

Slično, ako bi bilo da je $\overline{BD} > BC \Rightarrow \gamma > \omega \overset{(*)}{\Rightarrow} \gamma > \alpha$
$\Rightarrow \overline{AB} > \overline{BC}$
#kontradikcija

Pretpostavka da je $\overline{DB} \geq \overline{CB}$ nas vodi u kontradikciju pa nije tačna. Prema tome $\overline{DB} < \overline{CB}$, q.e.d.

Teorem (hipotenuza - ugao, HU)

U neutralnoj geometriji, neka su $\triangle ABC$ i $\triangle DEF$ pravougli trouglovi sa pravim uglom na vrhovima C i F. Ako je $\overline{AB} \cong \overline{DE}$ i $\measuredangle A \cong \measuredangle D$ tada je

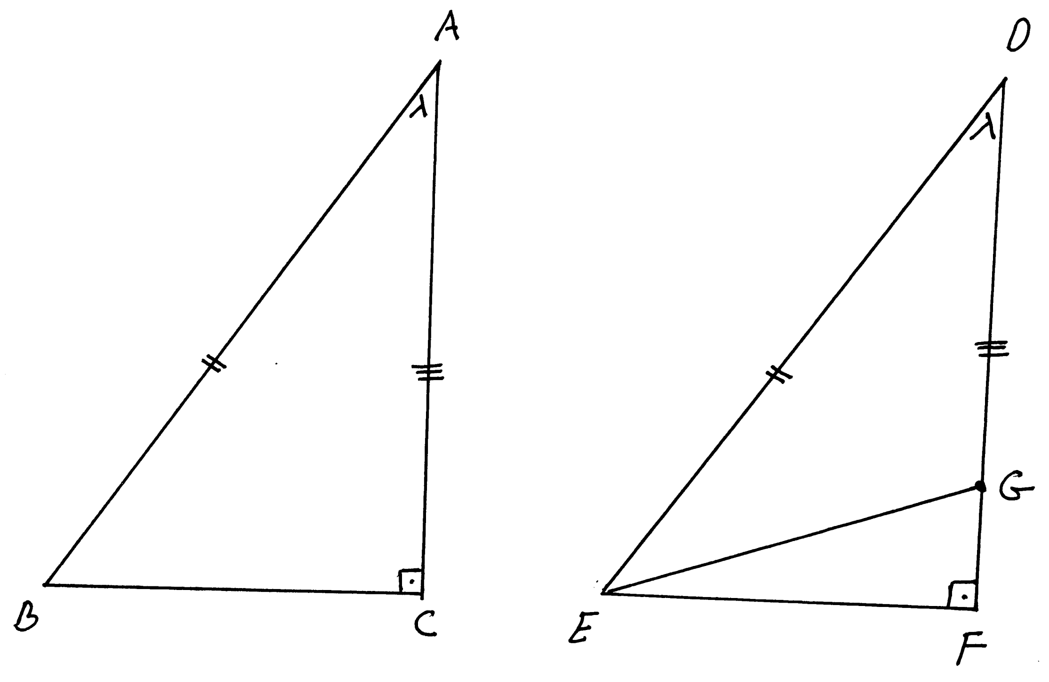
$$\triangle ABC \cong \triangle DEF.$$

(#) Dokazati teorem iznad

Rj. Ako $\overline{AC} \ncong \overline{DF}$ tada je jedna od ove dvije duži kraća od druge. Pretpostavimo da je $\overline{AC} < \overline{DF}$. Tada $\exists G$ t.d. $D-G-F$ i $\overline{AC} \cong \overline{DG}$. Sad

$$\left.\begin{array}{l} \overline{AB} \cong \overline{DE} \\ \measuredangle BAC \cong \measuredangle EDG \\ \overline{AC} \cong \overline{DG} \end{array}\right\} \overset{SUS}{\Longrightarrow} \triangle ABC \cong \triangle DEG \Rightarrow \measuredangle ACB \cong \measuredangle DGE \Rightarrow m(\measuredangle DGE) = 90$$

#kontradikcija ($\measuredangle DGE > \measuredangle DFE$ vanjski ugao)



Prema tome mora vrijediti da je $AC \cong DF$ pa možemo iskoristiti teoremu SUS i pokazati da $\triangle ABC \cong \triangle DEF$

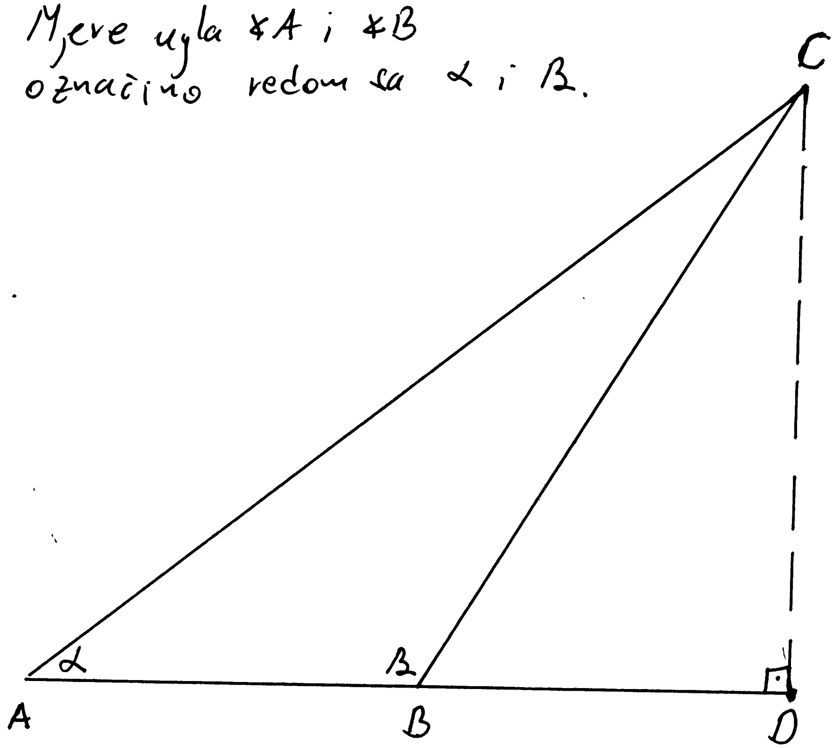
II način

Na trouglove $\triangle ABC$ i $\triangle DEF$ možemo iskoristiti pravilo SUU.

U neutralnoj geometriji, ako je tačka D podnožje visine trougla $\triangle ABC$ iz vrha C i ako je $A-B-D$, pokazati da je tada $\overline{CA} > \overline{CB}$.

Rj.

Mjere ugla $\measuredangle A$ i $\measuredangle B$ označimo redom sa α i β.



Primjetimo da je $\measuredangle ABC$ vanjski ugao $\triangle BDC$, pa je $\measuredangle ABC > \measuredangle BDC$ tj.

$$\beta > 90.$$

S obzirom da u $\triangle ABC$ možemo imati samo jedan tup ugao to je α oštar ugao pa je $\beta > \alpha$

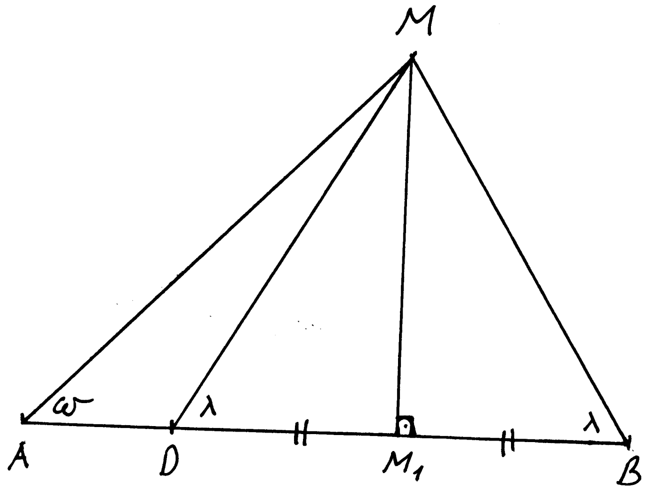
$$\beta > \alpha \Rightarrow \overline{AC} > \overline{BC}$$

q.e.d.

\# Označimo sa M_1 ortogonalnu projekciju tačke M na pravu određenu tačkama A i B t.d. vrijedi poredak $A-M_1-B$. Pokazati da je $\overline{MA} > \overline{MB}$ ako i samo ako $\overline{M_1A} > \overline{M_1B}$.

Rj:

„$\Leftarrow$" Pretpostavimo da je $M_1A > M_1B$. (i dokažimo da je $MA > MB$)



Tada $\exists D$ t.d. $B-M_1-D$ i $M_1D \cong M_1B$.

$$\left.\begin{array}{l} \overline{MM_1} \cong \overline{MM_1} \\ \sphericalangle MM_1D \cong \sphericalangle MM_1B \\ \overline{M_1D} \cong \overline{M_1B} \end{array}\right\} \overset{SUS}{\Longrightarrow} \Delta MM_1D \cong \Delta MM_1B$$

$$\Downarrow$$

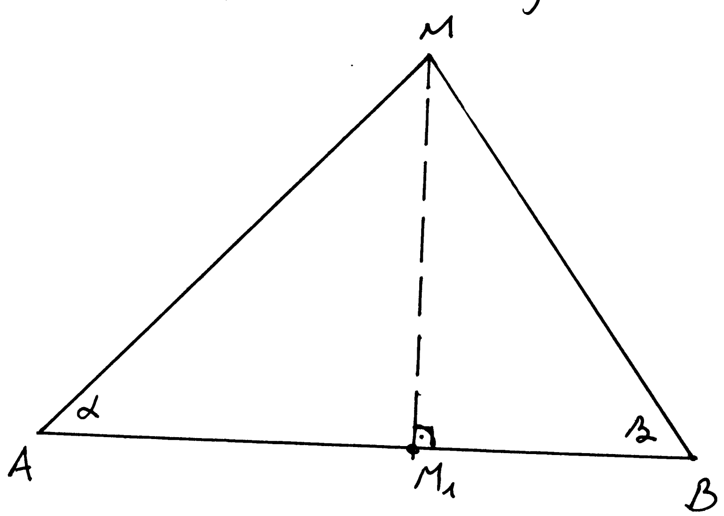
$$\sphericalangle MDM_1 \cong \sphericalangle MBM_1$$

Primjetimo da je $\sphericalangle MDM_1$ vanjski ugao ΔMAD pa je

$$\sphericalangle MAD < \sphericalangle MDM_1 \cong \sphericalangle MBM_1 \Rightarrow \sphericalangle A < \sphericalangle B \Rightarrow AM > BM$$

q.e.d.

„$\Rightarrow$" Pretpostavimo da je $\overline{AM} > \overline{BM}$ (i pokažimo da je tad $\overline{AM_1} > \overline{BM_1}$)



Označimo mjere uglova $\sphericalangle MAB$ i $\sphericalangle MBA$ redom sa α i β.

$\overline{AM} > \overline{MB} \Rightarrow \beta > \alpha$

Neka je D tačka na polupravoj $pp[M_1, A) = \overrightarrow{M_1A}$ takva da $\overline{M_1D} \cong \overline{M_1B}$.

Moguć je tačno jedan od sljedeća tri slučaja

1° M_1-A-D

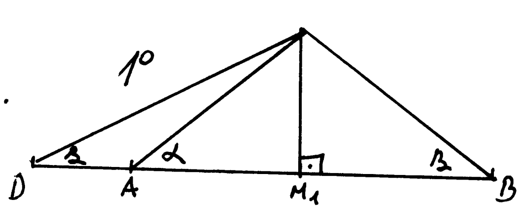
2° $D = A$

3° M_1-D-A.

Pokažimo da slučajevi 1° i 2° nisu mogući

ZAVRŠITI ZA VJEŽBU

1°



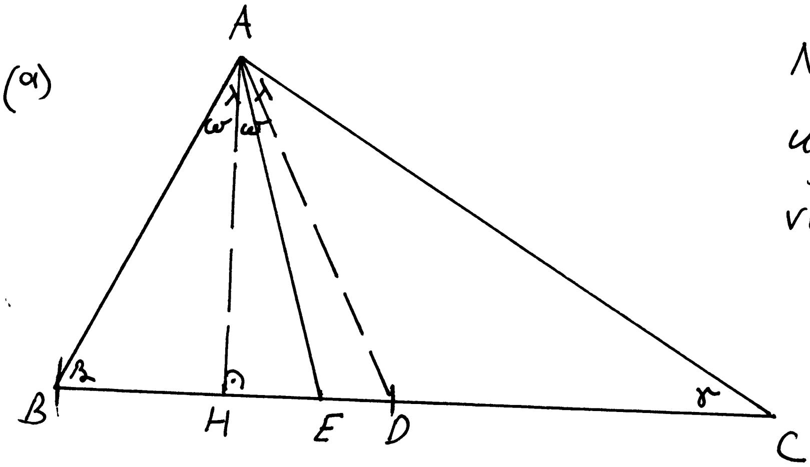
(#) U trouglu ΔABC je $\overline{AB} < \overline{AC}$. Neka su E, D i H redom tačke u kojima simetrala ugla, težišna linija i visina iz tjemena A sijeku pravu $\overleftrightarrow{BC}$ (pretpostavimo da je $B-H-C$, $B-E-C$ i $B-D-C$). Dokazati da vrijedi

(a) $\measuredangle AEB < \measuredangle AEC$;

(b) $\overline{BE} < \overline{CE}$;

(c) da je poredak $H-E-D$.

Rj. (a)



Neka je $\overrightarrow{AE}$ simetrala ugla $\measuredangle A$ i označimo sa $\overline{AH}$ visinu (iz vrha A)

Kako je $\overline{AB} < \overline{AC}$ prema prethodnom zadatku $\overline{BH} < \overline{CH}$.

Želimo pokazati da je poredak $B-H-E-C$.

Pa neka je D tačka na pp$[H,E) = \overrightarrow{HE}$ t.d. $BH \cong HD$ i $H-D-C$.

$$\left.\begin{array}{r} \overline{BH} \cong \overline{HD} \\ \measuredangle BHA \cong \measuredangle DHA \\ \overline{AH} \cong \overline{AH} \end{array}\right\} \overset{SUS}{\Longrightarrow} \Delta ABH \cong \Delta ADH$$

$$\Downarrow$$

$\measuredangle BAH \cong \measuredangle DAH$ (označimo mjere ovih uglova sa ω)

Ako označimo mjere uglova $\measuredangle BAE$ i $\measuredangle EAC$ sa λ, primjetimo da je

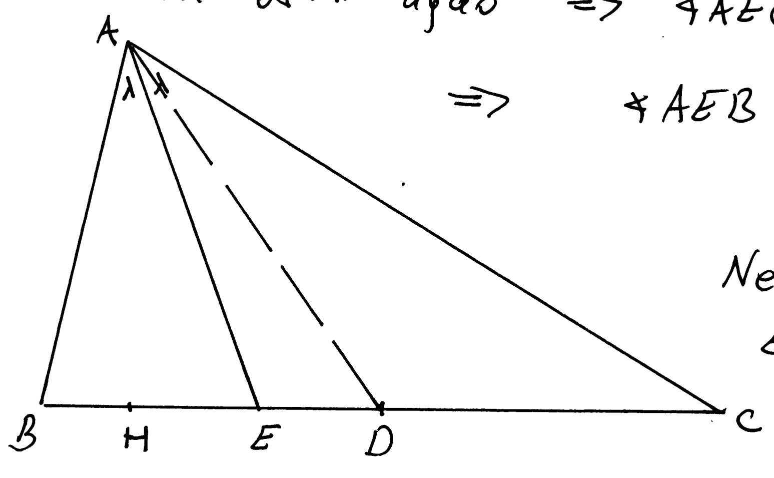
$2\omega < 2\lambda \Rightarrow \omega < \lambda \Rightarrow H \in int(\measuredangle BAE) \Rightarrow B-H-E$.

Sad posmatrajmo ΔAHE. Kako je $\measuredangle AHE$ prav ugao to $\measuredangle AEH$ mora biti oštar ugao $\Rightarrow \measuredangle AEC$ tup ugao

$\Rightarrow \measuredangle AEB < \measuredangle AEC$

q.e.d.

(b)

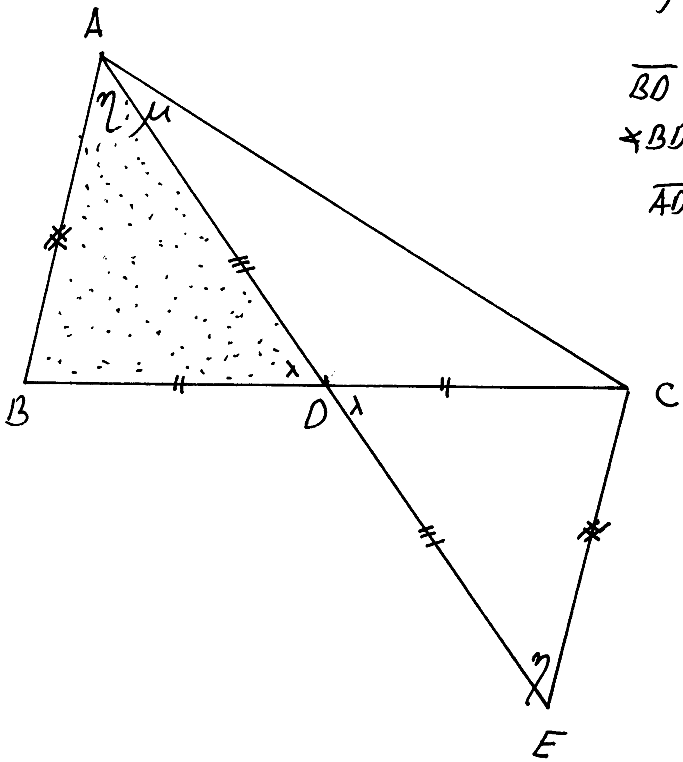


Neka je $\overrightarrow{AE}$ simetrala ugla $\measuredangle A$ t.d. $E \in \overline{BC}$ i pokažimo da je $\overline{BE} < \overline{CE}$.

Označimo sa 2λ mjeru ugla $\measuredangle A$.

Posmatrajmo težišnicu $\overline{AD}$ i pokažimo da je $\measuredangle BAD > \measuredangle DAC$.

Neka je $E \in \overrightarrow{AD}$ t.d. $A-D-E$; $\overline{DE} \cong \overline{AD}$



$$\left.\begin{array}{l} \overline{BD} \cong \overline{CD} \\ \measuredangle BDA \cong \measuredangle CDE \\ \overline{AD} \cong \overline{ED} \end{array}\right\} \overset{SUS}{\Longrightarrow} \triangle ABD \cong \triangle ECD$$

$$\Downarrow$$

$$\measuredangle BAD \cong \measuredangle CED$$

$$\text{i } \overline{AB} \cong \overline{EC}$$

Posmatrajmo $\triangle AEC$.

Kako je $\overline{AB} < \overline{AC}$ to je

$$\overline{EC} < \overline{AC} \Rightarrow \measuredangle DAC < \measuredangle DEC$$

$$\Downarrow$$

$$\measuredangle BAD > \measuredangle DAC$$

$$(\eta > \mu)$$

Kako je $\measuredangle BAD > \measuredangle CAD$ i $\measuredangle BAD + \measuredangle CAD = 2\lambda$ to je

$$\measuredangle BAD > \lambda \quad \Rightarrow \quad D \in \operatorname{int}(\measuredangle EAC)$$

$$\Downarrow$$

$$B-E-D-C$$

D sredina $\overline{BC}$ $\Rightarrow$ $BE < CE$

q.e.d

(c)

$$\left.\begin{array}{l} \text{Na osnovu (a) imamo } B-H-E-C \\ \text{Na osnovu (b) imamo } B-E-D-C \end{array}\right\} \Rightarrow H-E-D$$

q.e.d

(#) Ako je M sredina duži $\overline{BC}$ tada duž $\overline{AM}$ nazivamo težišnica trougla $\triangle ABC$.

(a.) Dokazati da u neutralnoj geometriji ako je $\triangle ABC$ jednakokraki trougao sa bazom $\overline{BC}$ tada je sljedeće kolinearno:

(i) težišnica iz tačke A;

(ii) simetrala ugla $\sphericalangle A$;

(iii) visina iz tačke A;

(iv) simetrala duži $\overline{BC}$.

(b.) Obrnuto, u neutralnoj geometriji dokazati da ako su bilo koje dvije od (i)-(iv) kolinearne tada je trougao jednakokraki (šest različitih slučajeva).

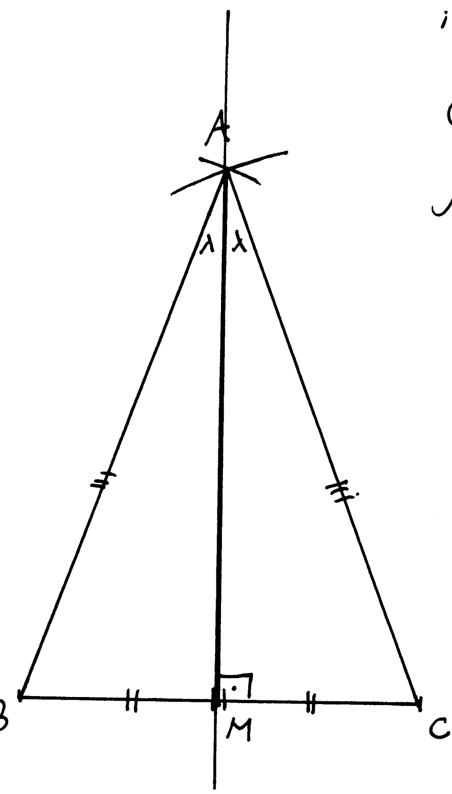
Rj. Prisjetimo se definicija

težišnica iz tačke A – duž $\overline{AM}$ (M sredina $\overline{BC}$).

simetrala ugla $\sphericalangle A$ – polupava p sa početnom tačkom A t.d. $\sphericalangle BAp \cong \sphericalangle CAp$.

visina iz tačke A – duž $\overline{AD}$ t.d. $D \in \overleftrightarrow{BC}$, $\overline{AD} \perp \overleftrightarrow{BC}$.

simetrala duži $\overline{BC}$ – prava koja prolazi kroz sredinu M duži $\overline{AB}$ i okomita je na $\overline{BC}$.



Označimo sa ℓ pravu koja sadrži težišnicu $\overline{AM}$ jkk $\triangle ABC$. Primjetimo da

$$\left.\begin{array}{l} \overline{BM} \cong \overline{CM} \\ \overline{AM} \cong \overline{AM} \\ \overline{AB} \cong \overline{AC} \end{array}\right\} \xRightarrow{SSS} \triangle AMB \cong \triangle AMC$$

$$\Downarrow$$

$$\sphericalangle BAM \cong \sphericalangle CAM \ ; \ \sphericalangle AMB \cong \sphericalangle AMC$$

$\sphericalangle BAM \cong \sphericalangle CAM \Rightarrow pp[AM) = \overrightarrow{AM}$ je simetrala $\sphericalangle A$

$\Rightarrow \overrightarrow{AM}$ pripada pravoj ℓ

$$m(\sphericalangle AMB) + m(\sphericalangle AMC) = 180 \xRightarrow{\sphericalangle AMB \cong \sphericalangle AMC} \sphericalangle AMB = 90 = \sphericalangle AMC$$

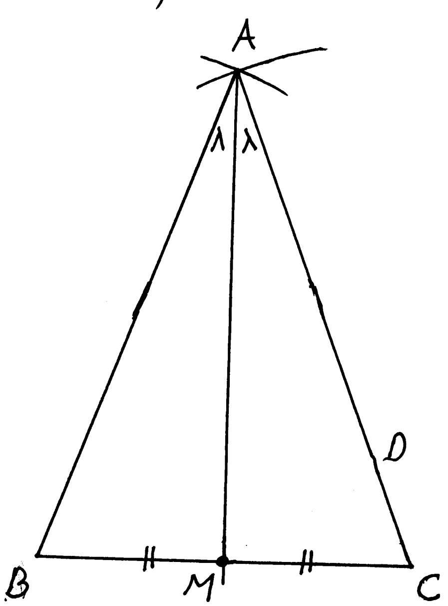
$\Rightarrow \overline{AM} \perp \overleftrightarrow{BC}$; $M \in \overleftrightarrow{BC}$ $\Rightarrow$ $\overline{AM}$ je visina

Primjetimo da $\overline{AM}$ pripada pravoj l.

Isto tako $l \perp \overline{BC}$, $M \in l \Rightarrow l$ je simetrala duži $\overline{BC}$

Time je težišnica iz A kolinearna sa simetralom $\measuredangle A$, sa simetralom duži $\overline{BC}$ i sa visinom iz A.

(b) Pretpostavimo da su težišnica iz A i simetrala $\measuredangle A$ kolinearni, i pokažimo da je trougao jednakokraki.



Označimo sa M sredinu BC.

Ako bi vrijedilo da je $\overline{AB} < \overline{AC}$ tada na osnovu prethodnog zadatka bi imali da je $\overline{BM} < \overline{CM}$

#kontradikcija

Za $\overline{AB} > \overline{AC}$ bi, na osnovu prethodnog zadatka imali da je $\overline{BM} > \overline{MC}$

#kontradikcija

Prema tome mora vrijediti da je $\overline{AB} \cong AC$.

$\Downarrow$

ΔABC jkk

q.e.d.

Nešto slično bi imali za ostalih pet slučajeva.

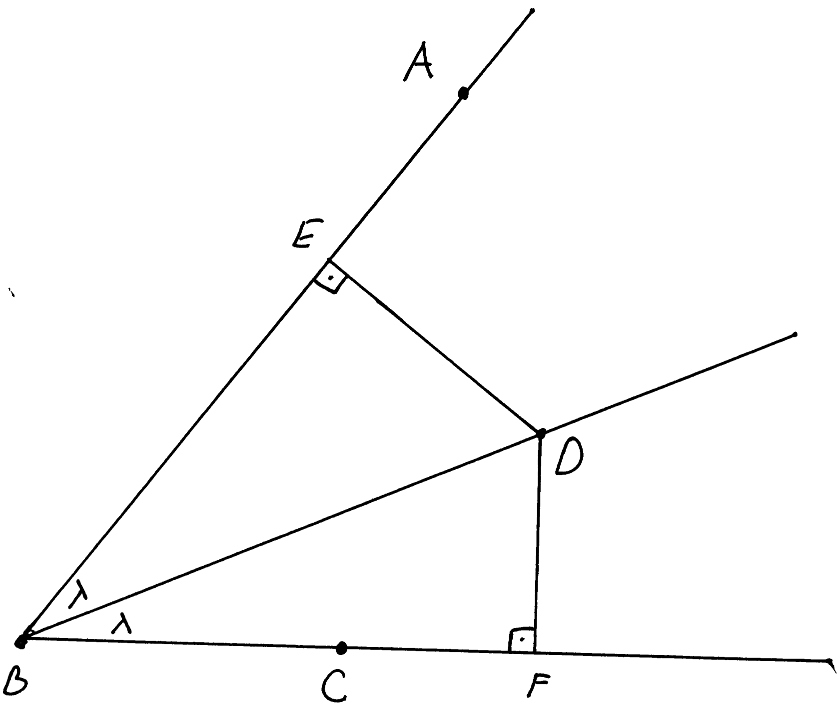
(ZA VJEŽBU)

Teorema

U neutralnoj geometriji, ako je $p[B,D) = \overrightarrow{BD}$ simetrala ugla $\measuredangle ABC$ i ako su E i F podnožja okomica iz tačke D na prave $p(B,A) = \overleftrightarrow{BA}$ i $p(B,C) = \overleftrightarrow{BC}$ tada je $\overline{DE} \cong \overline{DF}$.

(#) Dokazati teoremu iznad.

Rj.



Kako je $\overrightarrow{BD}$ simetrala ugla to je $\measuredangle EBD \cong \measuredangle DBF$.

Kako su E i F podnožja okomica to su $\measuredangle DEB$ i $\measuredangle DFB$ pravi uglovi, pa je $\measuredangle DEB \cong \measuredangle DFB$.

Sad imamo

$$\left.\begin{array}{l} \overline{BD} \cong \overline{BD} \\ \measuredangle DBE \cong \measuredangle DBF \\ \measuredangle DEB \cong \measuredangle DFB \end{array}\right\} \xRightarrow{SUU} \triangle DBE \cong \triangle DBF$$

$$\Downarrow$$

$$\overline{DE} \cong \overline{DF}$$

q.e.d.